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# Detecting topological transitivity of piecewise monotone interval maps <sup>☆</sup>

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## Abstract

Through kneading theory, developed by Milnor and Thurston, we present an algorithm which enables us to detect the topological transitivity of a relevant class of piecewise monotone interval maps.

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## 1. Introduction and statements of results

Kneading theory, introduced by Milnor and Thurston in [4], is a powerful tool to describe the qualitative behavior of successive iterates of piecewise monotone maps of the interval. In particular, it was shown in that paper that important topological invariants, such as topological entropy  $h_{\text{top}}(f)$  and the Artin–Mazur zeta function  $\zeta_f(t)$ , can be computed in terms of the kneading determinant. Our aim in this paper is to show that the same theory can be useful for detecting the topological transitivity of a piecewise monotone interval

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map. More precisely, we present an algorithm which enables us to identify the topological transitive maps of a relevant class of piecewise monotone maps of a compact interval  $I \subset \mathbb{R}$ .

Let us begin by introducing the basic notions.

**Definition 1.** A continuous map  $f: I \rightarrow I$  is topologically transitive if for any two non-empty open sets  $U, V \subset I$  there exists  $n \geq 0$  such that  $f^n(U) \cap V \neq \emptyset$ .

A continuous map  $f: I = [a, b] \rightarrow I$  is called  $\ell$ -modal if there are points  $a < c_1 < \dots < c_\ell < b$  at which  $f$  has a local extremum and such that  $f$  is strictly monotone in each of the connected components of  $I \setminus \{c_1, \dots, c_\ell\}$ . In this case the points  $c_1, \dots, c_\ell$  are called the turning points of  $f$ , the intervals  $[a, c_1], [c_1, c_2], \dots, [c_\ell, b]$  are called the laps of  $f$ , and the number  $\ell(f) = \ell + 1$  is called the lap number of  $f$ . A map  $f: I \rightarrow I$  is piecewise monotone if it is  $\ell$ -modal for some positive integer  $\ell$ . In what follows we shall use the notation  $\mathcal{M}(I)$  to denote the set of all piecewise monotone maps on  $I$ . The set of all topologically transitive maps  $f \in \mathcal{M}(I)$  will be denoted by  $\mathcal{T}(I)$ .

Let  $f \in \mathcal{M}(I)$ , the  $n$ th-iterate of  $f$  is by definition the map  $f^n \in \mathcal{M}(I)$ , defined by

$$f^n = \overbrace{f \circ \dots \circ f}^{n \times}.$$

As usually, for a given  $x \in I$ , the orbit of  $x$  under the action of  $f$  is the set  $\{f^n(x): n \in \mathbb{N}\}$ . The growth number of  $f$  is defined by

$$s = \lim_{n \rightarrow \infty} \sqrt[n]{\ell(f^n)}, \quad (1)$$

and the topological entropy of  $f$  will be denoted by  $h_{\text{top}}(f)$ . As a consequence of the main relationship

$$h_{\text{top}}(f) = \log(s), \quad (2)$$

due to Misiurewicz and Szlenk [5], we can write

$$\mathcal{M}^+(I) = \{f \in \mathcal{M}(I): s > 1\},$$

where  $\mathcal{M}^+(I)$  denote the set of all maps  $f \in \mathcal{M}(I)$  with  $h_{\text{top}}(f) > 0$ .

The relationship between topological entropy and topological transitivity has been studied by several authors. In particular, Blokh proved in [3] that

$$h_{\text{top}}(f) \geq (1/2) \log 2$$

holds for any topologically transitive map of the interval. So, we have

$$\mathcal{T}(I) \subset \mathcal{M}^+(I).$$

Among piecewise monotone maps there exists an important class of maps. A map  $f \in \mathcal{M}(I)$  is said to be  $r$ -piecewise linear if there exists a real number  $r > 0$  such that  $f(y) - f(x) = \pm r(y - x)$ , for all  $x$  and  $y$  lying in a same lap of  $f$ .

**Definition 2.** Let  $f \in \mathcal{M}^+(I)$ . A continuous, onto and increasing (not necessarily strictly increasing) map  $h: I \rightarrow I$  is said to be an  $s$ -semiconjugacy of  $f$  if there exists an  $s$ -piecewise linear map  $g: I \rightarrow I$  such that  $h \circ f = g \circ h$ .

Recall that Parry [6], and later Milnor and Thurston [4], showed that for any  $f \in \mathcal{M}^+(I)$  there exists at least one  $s$ -semiconjugacy of  $f$ . Furthermore, if  $f$  is topologically transitive then the mentioned  $s$ -semiconjugacy is in fact an homeomorphism (see Preston [5]). Thus, denoting by  $\mathcal{S}(f)$  the set of all  $s$ -semiconjugacies of  $f$ , and defining

$$\mathcal{L}(I) = \{f \in \mathcal{M}^+(I) : \text{there exists an homeomorphism } h \in \mathcal{S}(f)\},$$

we can write

$$\mathcal{T}(I) \subset \mathcal{M}^+(I) \cap \mathcal{L}(I) \subset \mathcal{L}(I).$$

So, from now on it makes sense to restrict the discussion to  $\mathcal{L}(I)$ . The main result in this article provides an algorithm which enables us to detect the topological transitivity of any  $f \in \mathcal{L}(I)$ . For any  $\ell$ -modal map  $f \in \mathcal{L}(I)$ , we will introduce a matrix

$$\mathbf{M}(f) \in \mathbb{R}^{\ell \times \ell},$$

and a family of matrices

$$\mathbf{M}(f; y) \in \mathbb{R}^{(\ell+1) \times (\ell+1)}, \quad \text{with } y \in I.$$

We will prove then that the study of the eigenvectors of  $\mathbf{M}(f)$ , and the eigenvalues of  $\mathbf{M}(f; y)$  play a relevant role in detecting the topological transitivity of  $f$ . Denoting by  $\sigma(\mathbf{A})$  the set of all eigenvalues (or *spectrum*) of a square matrix  $\mathbf{A}$ , the main result of this paper can be stated as follows:

**Theorem 3.** *Let  $f \in \mathcal{L}(I)$ . If  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1$  and  $s \notin \sigma(\mathbf{M}(f; y))$ , for all  $y$  lying in some dense subset of  $I$ , then  $f \in \mathcal{T}(I)$ .*

Since, for any  $f \in \mathcal{L}(I)$ , it is possible to calculate  $\mathbf{M}(f)$  and  $\mathbf{M}(f; y)$ , for all  $y \in I$ , with any intended precision, Theorem 3 supplies an algorithm which allows us to detect the topological transitivity of such maps.

Recall that a map  $f \in \mathcal{M}(I)$  is said to be Markov if the orbits of its turning points are finite. As one knows, if  $f \in \mathcal{T}(I)$  is Markov, then the corresponding transition matrix is irreducible, and we can use this to prove the following:

**Theorem 4.** *Let  $f \in \mathcal{T}(I)$  be a Markov map. Then  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1$  and  $s \notin \sigma(\mathbf{M}(f; y))$ , for all  $y \in I$ .*

We believe that Theorem 4 holds for all  $f \in \mathcal{T}(I)$ . However, some technical difficulties prevent us from presenting a general proof. Nevertheless we think that Theorems 3 and 4 justify the following conjecture:

**Conjecture 5.** *Let  $f \in \mathcal{L}(I)$ . Then  $f \in \mathcal{T}(I)$  if and only if  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1$  and  $s \notin \sigma(\mathbf{M}(f; y))$ , for all  $y$  lying in some dense subset of  $I$ .*

## 2. Proofs

The proof of Theorem 3 uses the kneading theory introduced by Milnor and Thurston in [4]. We will follow the kneading theory approach presented in [1,2].

First we recall some algebraic notions and definitions needed in the sequel. Let  $V$  be a vector space over  $\mathbb{R}$  and let  $\varphi: V \rightarrow V$  be a linear map with finite rank. As usually we define the trace of  $\varphi$  by

$$\mathrm{tr}(\varphi) = \mathrm{tr}(\varphi|_{\varphi(V)}).$$

If  $\varphi$  has finite rank, then there are vectors  $v_1, \dots, v_k \in V$  and linear forms  $\xi_1, \dots, \xi_k \in V^*$  such that

$$\varphi = \sum_{i=1}^k \xi_i \otimes v_i.$$

Considering the matrix  $\mathbf{M} = [\mathbf{m}_{i,j}] \in \mathbb{R}^{k \times k}$ , defined by

$$\mathbf{m}_{i,j} = \xi_j(v_i), \quad (3)$$

we have

$$\mathrm{tr}(\varphi) = \mathrm{tr}(\mathbf{M}).$$

More generally, if  $\varphi$  has finite rank then, for each  $n \geq 1$ ,  $\varphi^n$  has finite rank and

$$\mathrm{tr}(\varphi^n) = \mathrm{tr}(\mathbf{M}^n).$$

The following result is well known and gives an explicit method to compute the numbers  $\mathrm{tr}(\varphi^n)$ , for  $n \geq 1$ . Defining the determinant of  $\varphi$  to be the following formal power series

$$D_\varphi(t) = \exp \sum_{n \geq 1} -\mathrm{tr}(\varphi^n) \frac{t^n}{n} \in \mathbb{R}[[t]],$$

we have:

$$D_\varphi(t) = \det(\mathbf{I} - t\mathbf{M}) \quad (4)$$

holds in  $\mathbb{R}[[t]]$ , where  $\mathbf{I}$  denotes the  $k \times k$ -identity matrix.

Now we consider a more general situation. By a pair of endomorphisms  $(\varphi_0, \varphi_1)$  on  $V$ , we mean two finite-codimensional subspaces  $V_0$  and  $V_1$  of the same  $\mathbb{R}$ -vector space  $V$  and two linear maps  $\varphi_0: V_0 \rightarrow V_0$  and  $\varphi_1: V_1 \rightarrow V_1$ .

**Definition 6.** We say that the pair of endomorphisms  $(\varphi_0, \varphi_1)$  on  $V$  has *finite rank* if there exist extensions  $\theta_i$  of  $\varphi_i$  to  $V$  such that  $\theta_1 - \theta_0$  has finite rank.

So, if the pair  $(\varphi_0, \varphi_1)$  has finite rank the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_j & \xrightarrow{\subseteq} & V & \xrightarrow{\mathrm{pr}} & V/V_j \longrightarrow 0 \\ & & \downarrow \varphi_j & & \downarrow \theta_j & & \downarrow \tilde{\theta}_j \\ 0 & \longrightarrow & V_j & \xrightarrow{\subseteq} & V & \xrightarrow{\mathrm{pr}} & V/V_j \longrightarrow 0 \end{array}$$

commutes for  $j = 0, 1$ , and we may define *trace* of the pair  $(\varphi_0, \varphi_1)$  by setting

$$\mathrm{tr}(\varphi_0, \varphi_1) = \mathrm{tr}(\theta_1 - \theta_0) - \mathrm{tr}(\tilde{\theta}_1) + \mathrm{tr}(\tilde{\theta}_0).$$

It is easy to see that the definition does not depend on  $\theta_1$  and  $\theta_0$ . Observe that an endomorphism  $\varphi: V \rightarrow V$  with finite rank can be regarded as a pair of endomorphisms with finite rank. Considering the pair  $(0, \varphi)$ , where  $0: V \rightarrow V$  denotes the zero map, we see that  $\varphi$  has finite rank if and only if the pair  $(0, \varphi)$  has finite rank, and  $\text{tr}(0, \varphi) = \text{tr}(\varphi)$ . More generally, if the linear maps  $\varphi_0$  and  $\varphi_1$  both have finite ranks then the pair  $(\varphi_0, \varphi_1)$  has also finite rank and

$$\text{tr}(\varphi_0, \varphi_1) = \text{tr}(\varphi_1) - \text{tr}(\varphi_0).$$

Of course, in general the single traces in the previous formula are not defined.

Let  $(\varphi_0, \varphi_1)$  be a pair of endomorphisms on  $V$  having finite rank, and consider endomorphisms  $\theta_0$  and  $\theta_1$  as in Definition 6. Since  $\theta_1 - \theta_0$  has finite rank, there are vectors  $v_1, \dots, v_k \in V$  and linear forms  $\xi_1, \dots, \xi_k \in V^*$  such that

$$\theta_1 - \theta_0 = \sum_{i=1}^k \xi_i \otimes v_i, \quad (5)$$

and more generally

$$\theta_1^n - \theta_0^n = \sum_{i=1}^k \sum_{j=1}^n (\xi_i \circ \theta_1^{n-j}) \otimes \theta_0^{j-1}(v_i),$$

for all  $n \geq 1$ . This shows that  $\theta_1^n - \theta_0^n$  has finite rank for each  $n \geq 1$ , and once more from Definition 6 we may conclude that the pair  $(\varphi_0^n, \varphi_1^n)$  has finite rank and

$$\text{tr}(\varphi_0^n, \varphi_1^n) = \text{tr}(\theta_1^n - \theta_0^n) - \text{tr}(\tilde{\theta}_1^n) + \text{tr}(\tilde{\theta}_0^n),$$

for all  $n \geq 1$ .

**Definition 7.** Let  $(\varphi_0, \varphi_1)$  be a pair of endomorphisms having finite rank. We define the *determinant* of  $(\varphi_0, \varphi_1)$  to be the following element of  $\mathbb{R}[[t]]$

$$D_{(\varphi_0, \varphi_1)}(t) = \exp - \sum_{n \geq 1} \text{tr}(\varphi_0^n, \varphi_1^n) \frac{t^n}{n}.$$

As a consequence of the definition we have:

**Proposition 8.** Let  $(\varphi_0, \varphi_1)$  be pair of endomorphisms on  $V$  having finite rank, then  $(\varphi_1, \varphi_0)$  has finite rank and  $D_{(\varphi_0, \varphi_1)}(t) = D_{(\varphi_1, \varphi_0)}(t)^{-1}$  holds in  $\mathbb{R}[[t]]$ . Furthermore, if  $(\varphi_1, \varphi_2)$  is another pair of endomorphisms on  $V$  having finite rank, then the pair  $(\varphi_0, \varphi_2)$  has also finite rank and  $D_{(\varphi_0, \varphi_2)}(t) = D_{(\varphi_0, \varphi_1)}(t)D_{(\varphi_1, \varphi_2)}(t)$  holds in  $\mathbb{R}[[t]]$ .

Observe that if  $\varphi$  has finite rank then

$$D_{(0, \varphi)}(t) = D_{\varphi}(t).$$

If  $\varphi_0$  and  $\varphi_1$  both have finite ranks then

$$D_{(\varphi_0, \varphi_1)}(t) = D_{\varphi_1}(t)D_{\varphi_0}(t)^{-1}. \quad (6)$$

So, in these cases, we can use Eq. (4) to compute  $D_{(\varphi_0, \varphi_1)}(t)$ . Obviously, in the general case, Eq. (4) does not allow us to compute  $D_{(\varphi_0, \varphi_1)}(t)$ — $D_{\varphi_0}(t)$  and  $D_{\varphi_1}(t)$  are not defined in general. In order to compute  $D_{(\varphi_0, \varphi_1)}(t)$  in the general case, we generalize Eq. (4). Let  $\theta_0$  and  $\theta_1$  be endomorphisms as in Definition 6. Considering vectors  $v_1, \dots, v_k \in V$  and linear forms  $\xi_1, \dots, \xi_k \in V^*$  as in Eq. (5), we define the matrix  $\mathbf{M}(t) = [\mathbf{m}_{i,j}(t)] \in \mathbb{R}[[t]]^{k \times k}$  by

$$\mathbf{m}_{i,j}(t) = \sum_{n \geq 0} \xi_j(\theta_0^n(v_i)) t^n \in \mathbb{R}[[t]]. \quad (7)$$

Observe that if we identify an endomorphism with finite rank  $\varphi: V \rightarrow V$  with the corresponding pair of finite rank  $(0, \varphi)$  then the matrix  $\mathbf{M}(t)$  coincides with the matrix  $\mathbf{M}$  defined in Eq. (3). Thus the next theorem, which gives an explicit method to compute  $D_{(\varphi_0, \varphi_1)}(t)$ , can be regarded as a generalization of Eq. (4).

**Theorem 9.** *Let  $(\varphi_0, \varphi_1)$  be a pair of endomorphisms having finite rank. Then*

$$D_{(\varphi_0, \varphi_1)}(t) = \det(\mathbf{I} - t\mathbf{M}(t)) D_{\tilde{\theta}_0}(t) D_{\tilde{\theta}_1}(t)^{-1}$$

*holds in  $\mathbb{R}[[t]]$ .*

Let  $\theta_0: V \rightarrow V$  and  $\theta_1: V \rightarrow V$  be linear maps, and suppose that there exist  $v_1, \dots, v_k \in V$  and  $\xi_1, \dots, \xi_k \in V^*$  verifying Eq. (5). From Theorem 9 we know that

$$D_{(\theta_0, \theta_1)}(t) = \det(\mathbf{I} - t\mathbf{M}(t)) \quad (8)$$

holds in  $\mathbb{R}[[t]]$ . Notice that if  $\xi_1, \dots, \xi_k \in B_{\theta_0}(\rho)$ , where  $B_{\theta_0}(\rho)$  is the subspace of  $V^*$  defined by

$$B_{\theta_0}(\rho) = \left\{ \xi \in V^*: \overline{\lim_{n \rightarrow \infty}} \sqrt[n]{|\xi(\theta_0^n(v))|} \leq \rho, \text{ for all } v \in V \right\}, \quad (9)$$

then the entries of  $\mathbf{M}(t)$ , and consequently  $D_{(\theta_0, \theta_1)}(t)$ , are convergent for all  $|t| < \rho^{-1}$ . So, for all  $|\lambda| > \rho$ , we have a matrix  $\mathbf{M}(\lambda^{-1}) = [\mathbf{m}_{i,j}(\lambda^{-1})] \in \mathbb{R}^{k \times k}$  defined by

$$\mathbf{m}_{i,j}(\lambda^{-1}) = \sum_{n \geq 0} \xi_j(\theta_0^n(v_i)) \lambda^{-n} \in \mathbb{R},$$

and from Eq. (8) we conclude:  $D_{(\theta_0, \theta_1)}(\lambda^{-1}) = 0$  if and only if  $\lambda$  is an eigenvalue of  $\mathbf{M}(\lambda^{-1})$ . It was shown in [2] that the eigenvectors of  $\mathbf{M}(\lambda^{-1})$  associated to  $\lambda$  are useful to compute the eigenvectors of the dual

$$\begin{aligned} \theta_1^*: V^* &\rightarrow V^*, \\ \xi &\rightarrow \xi \circ \theta_1. \end{aligned}$$

In fact, Theorem 2.4 of [2] shows that, for any  $\xi \in B_{\theta_0}(\rho)$  we have:

$$\xi \in E_{\theta_1^*}(\lambda) = \{ \xi \in V^*: \xi \circ \theta_1 = \lambda \xi \}$$

if and only if there exists a unique

$$(a_1, \dots, a_k) \in \ker(\mathbf{M}(\lambda^{-1}) - \lambda \mathbf{I})$$

such that  $\xi = a_1\eta_1 + \cdots + a_k\eta_k$ , where  $\eta_i \in V^*$  is defined by

$$\eta_i(v) = \sum_{n \geq 0} \xi_i(\theta_0^n(v))\lambda^{-n}, \quad \text{for all } v \in V.$$

So, as an immediate consequence of this we obtain:

**Theorem 10.** *Let  $\theta_0: V \rightarrow V$  and  $\theta_1: V \rightarrow V$  be linear maps. Suppose that there exist  $v_1, \dots, v_k \in V$  and  $\xi_1, \dots, \xi_k \in B_{\theta_0}(\rho)$  verifying Eq. (5). Then we have*

$$\dim B_{\theta_0}(\rho) \cap E_{\theta_1^*}(\lambda) = \dim \ker(\mathbf{M}(\lambda^{-1}) - \lambda \mathbf{I}),$$

for all  $|\lambda| > \rho$ .

Notice that, if  $V$  is finite-dimensional, and  $r(\theta_0)$  denotes the spectral radius of  $\theta_0$ , then  $B_{\theta_0}(r(\theta_0)) = V^*$ , and consequently:

**Corollary 11.** *Let  $\theta_0: V \rightarrow V$  and  $\theta_1: V \rightarrow V$  be linear maps on a finite-dimensional vector space  $V$ . If  $v_1, \dots, v_k \in V$  and  $\xi_1, \dots, \xi_k \in V^*$  verify Eq. (5), then we have*

$$\dim E_{\theta_1^*}(\lambda) = \dim \ker(\mathbf{M}(\lambda^{-1}) - \lambda \mathbf{I}),$$

for all  $|\lambda| > r(\theta_0)$ .

### 2.1. The matrix $M(f)$

Let  $X$  be an arbitrary (finite or infinite) set. In what follows  $S_0(X)$  denotes the  $\mathbb{R}$ -vector space whose basis are the formal symbols  $x \in X$ . We denote by  $S_1(X)$  the subspace of  $S_0(X)$  which is generated by the vectors  $y - x$ , with  $y, x \in X$ . If  $f: X \rightarrow X$  is a map, we denote by  $f_{\#0}: S_0(X) \rightarrow S_0(X)$  the unique linear map that verifies  $f_{\#0}(x) = f(x)$ , for all  $x \in X$ .

Let  $f \in \mathcal{M}(I)$ , this map induces the sign

$$\varepsilon: I \rightarrow \{-1, 0, 1\},$$

defined as follows: if  $x \in I$  is not a turning point of  $f$ , we define  $\varepsilon(x) = \pm 1$  according as  $f$  is strictly increasing or strictly decreasing on some neighborhood of  $x$ ; if  $x$  is a turning point of  $f$  we define  $\varepsilon(x) = 0$ . If  $J \subseteq I$  is an interval on which  $f$  is monotone we also define  $\varepsilon(J) = \pm 1$  according as  $f$  is strictly increasing or strictly decreasing on  $J$ . Notice that  $S_1(I)$  and  $S_0(\text{int}(I))$  are finite-codimensional subspaces of  $S_0(I)$ . Indeed

$$\dim S_0(I)/S_1(I) = 1 \quad \text{and} \quad \dim S_0(I)/S_0(\text{int}(I)) = 2.$$

Furthermore, since  $f$  is piecewise monotone, the set

$$\mathcal{I}_f = \{y - x \in S_1(I): f \text{ is monotone on } [x, y]\}$$

spans  $S_1(I)$ . With this, we have everything that we needed to define the pair  $(\varepsilon f_{\#0}, \varepsilon f_{\#1})$  of linear endomorphisms on  $S_0(I)$ .

**Definition 12.** Let  $f \in \mathcal{M}(I)$ . Define  $\varepsilon f_{\#0}$  to be the unique linear endomorphism of  $S_0(\text{int}(I))$  that verifies:  $\varepsilon f_{\#0}(x) = \varepsilon(x) f_{\#0}(x)$  if  $x \in \text{int}(I)$ . Define  $\varepsilon f_{\#1}$  to be the unique linear endomorphism of  $S_1(I)$  that verifies:  $\varepsilon f_{\#1}(y - x) = \varepsilon([x, y]) f_{\#0}(y - x)$ , for all  $y - x \in \mathcal{I}_f$ .

Thus, for each  $f \in \mathcal{M}(I)$ , we have a pair of linear endomorphisms  $(\varepsilon f_{\#0}, \varepsilon f_{\#1})$  on  $S_0(I)$ . Next we prove that this pair has finite rank. For this purpose we need first to define extensions of  $\theta_0$  and  $\theta_1$  to the common superspace  $S_0(I)$ .

Let  $f \in \mathcal{M}(I)$ . For each  $y \in I$ , define the step function  $\alpha_y : I \rightarrow \{-1, 0, 1\}$  by setting

$$\alpha_y(x) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x = y, \\ -1 & \text{if } x < y. \end{cases} \quad (10)$$

If  $c_j$  is a turning point of  $f$ , define  $\omega_j : S_0(I) \rightarrow \mathbb{R}$  to be the unique linear form of  $S_0(I)$  verifying:

$$\omega_j(x) = \varepsilon(c_j -)(k_j + \alpha_{c_j}(x)), \quad \text{for all } x \in I, \quad (11)$$

with  $k_1 = 0$ , and  $k_j = 2$  for  $2 \leq j \leq \ell$ . Using these linear forms, we define  $\theta_1 : S_0(I) \rightarrow S_0(I)$  by

$$\theta_1 - \theta_0 = \sum_{j=1}^{\ell} \omega_j \otimes v_j, \quad (12)$$

with  $v_j = f_{\#0}(c_j) \in S_0(I)$ , and where  $\theta_0 : S_0(I) \rightarrow S_0(I)$  is the unique linear endomorphism that verifies:

$$\theta_0(x) = \varepsilon(x) f_{\#0}(x), \quad \text{for all } x \in I.$$

Notice that  $\theta_1$  is the unique extension of  $\varepsilon f_{\#1}$  to  $S_0(I)$  that verifies  $\theta_1(c_1) = 0$ , and, obviously,  $\theta_0$  is an extension of  $\varepsilon f_{\#0}$ . Thus, from Eq. (12), it follows that  $(\varepsilon f_{\#0}, \varepsilon f_{\#1})$  has finite rank, and so we may define the kneading determinant of  $f$ ,  $\mathbf{D}(t) \in \mathbb{R}[[t]]$ , by

$$\mathbf{D}(t) = D_{(\varepsilon f_{\#0}, \varepsilon f_{\#1})}(t).$$

Notice that since  $\theta_1(c_1) = 0$ , it follows  $\tilde{\theta}_1 = 0$ . Thus, from Eq. (12) and Theorem 9, we see that

$$\mathbf{D}(t) = \det(\mathbf{I} - t \mathbf{M}(t)) D_{\tilde{\theta}_0}(t) \quad (13)$$

holds in  $\mathbb{R}[[t]]$ , where  $\mathbf{M}(t) = [\mathbf{m}_{i,j}(t)] \in \mathbb{R}[[t]]^{\ell \times \ell}$  is defined by<sup>1</sup>

$$\mathbf{m}_{i,j}(t) = \sum_{n \geq 0} \omega_j(\theta_0^n(f_{\#0}(c_i))) t^n \in \mathbb{R}[[t]]. \quad (14)$$

<sup>1</sup> The extensions  $\theta_0$  and  $\theta_1$  that we defined here are different from the ones that we used in [1,2]. Let us notice that  $\mathbf{D}(t)$  does not depend on these extensions. However the definition of  $\mathbf{M}(t)$  depends on  $\theta_0$  and  $\theta_1$ . With this choice we obtain a matrix  $\mathbf{M}(f) \in \mathbb{R}^{\ell \times \ell}$ , with the other extensions we would have a matrix lying in  $\mathbb{R}^{(\ell+1) \times (\ell+1)}$ , making it more difficult to calculate the eigenvectors of  $\mathbf{M}(f)$ .



The entries of the kneading matrix  $\mathbf{M}(t)$  are formal power series that can be computed in of the orbits of points  $f(c_1), \dots, f(c_\ell)$ . It is easy to show that these entries are convergent for all  $|t| < 1$ . Indeed, if we define

$$B(\rho) = \{\omega \in S_0(I)^*: |\omega(x)| \leq \rho, \text{ for all } x \in I\},$$

then

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|\omega(\theta_0^n(x))|} \leq \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|\omega(f_{\#0}^n(x))|} \leq \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\rho} = 1,$$

for all  $\omega \in B(\rho)$ . So, according to the definition given in Eq. (9), we have

$$B(\rho) \subseteq B_{\theta_0}(1), \quad \text{for all } \rho \in \mathbb{R}^+, \quad (15)$$

and by the definition of  $\omega_j$

$$\omega_j \in B(2) \subseteq B_{\theta_0}(1), \quad \text{for all } 1 \leq j \leq \ell. \quad (16)$$

Thus, for any  $f \in \mathcal{M}(I)$  and  $|\lambda| > 1$ , we have a matrix

$$\mathbf{M}(\lambda^{-1}) = [\mathbf{m}_{i,j}(\lambda^{-1})] \in \mathbb{R}^{\ell \times \ell}.$$

**Proposition 13.** *Let  $f \in \mathcal{M}(I)$ . Then the formal power series  $\mathbf{D}(t)$  converges for all  $|t| < 1$ . Furthermore, if  $|\lambda| > 1$  then  $\mathbf{D}(\lambda^{-1}) = 0$  if and only if  $\lambda$  is an eigenvalue of  $\mathbf{M}(\lambda^{-1})$ .*

**Proof.** Let  $r(\tilde{\theta}_0)$  be the spectral radius of  $\tilde{\theta}_0$ . By the definition of  $\theta_0$ , we have  $|\text{tr}(\tilde{\theta}_0^n)| \leq 2$ , for all  $n \geq 1$ , therefore  $r(\tilde{\theta}_0) = \overline{\lim}_n |\text{tr}(\tilde{\theta}_0^n)|^{1/n} \leq 1$ . Consequently, if  $|\lambda| > 1$ , then  $\lambda^{-1}$  is not a root of the polynomial  $\mathbf{D}_{\tilde{\theta}_0}(t)$ . Thus, from Eq. (13), we see that  $\mathbf{D}(t)$  converges for all  $|t| < 1$ , and  $\mathbf{D}(\lambda^{-1}) = 0$  if and only if  $\det(\mathbf{I} - \lambda^{-1}\mathbf{M}(\lambda^{-1})) = \lambda^{-\ell} \det(\lambda\mathbf{I} - \mathbf{M}(\lambda^{-1})) = 0$ , as desired.  $\square$

We have now everything that is necessary to define the matrix  $\mathbf{M}(f)$ . Notice that, if  $f \in \mathcal{M}^+(I)$ , we have  $s > 1$ , and therefore one can define the matrix  $\mathbf{M}(f) \in \mathbb{R}^{\ell \times \ell}$  by setting

$$\mathbf{M}(f) = \mathbf{M}(s^{-1}).$$

Recall that Milnor and Thurston proved that: if  $f \in \mathcal{M}^+(I)$ , then  $s^{-1}$  is the first zero in  $]0, 1[$  of  $\mathbf{D}(t)^2$ . So, we have

$$\mathbf{D}(s^{-1}) = 0, \quad (17)$$

<sup>2</sup> The kneading determinant  $D(t)$  introduced by Milnor and Thurston does not coincide with  $\mathbf{D}(t)$ . But from Remark 2.25 of [1] we have

$$\mathbf{D}(t) = (1 - t^{p_1}) \cdots (1 - t^{p_m}) D(t),$$

where  $p_1, \dots, p_m$  are the periods of the periodic orbits of  $f$  which intersect  $\{a, c_1, \dots, c_\ell, b\}$ . Thus, if  $|\lambda| > 1$ , we have  $\mathbf{D}(\lambda^{-1}) = 0$  if and only if  $\mathbf{D}(\lambda^{-1}) = 0$ .

and from Proposition 13

$$s \in \sigma(\mathbf{M}(f)). \quad (18)$$

We already said that for any  $f \in \mathcal{M}^+(I)$  there exists at least a  $s$ -semiconjugacy of  $f$ . We will see later that the set of all  $s$ -semiconjugacies of a map  $f \in \mathcal{L}(I)$  plays a relevant role in the study of the topological transitivity of  $f$ . On the other hand it was shown in [2] that this set can be characterized in terms of the eigenvectors of  $\mathbf{M}(f)$  associated to  $s$ . For our purposes the following result will be sufficient.

**Theorem 14.** *Let  $f \in \mathcal{M}^+(I)$ . If  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1$ , then there exists one and only one  $s$ -semiconjugacy of  $f$ .*

**Proof.** Let  $h_1 : I = [a, b] \rightarrow I$  and  $h_2 : I \rightarrow I$  be two  $s$ -semiconjugacies of  $f$ . For each  $i = 1, 2$ , define a linear form  $\xi_i \in S_1(I)^*$  by setting  $\xi_i(x - y) = h_i(x) - h_i(y)$ , for all  $x, y \in I$ . Notice that, since  $h_i(a) = a$  and  $h_i(b) = b$ , we have  $h_1 = h_2$  if and only if  $\xi_1$  and  $\xi_2$  are linearly dependent. In order to prove that  $\xi_1$  and  $\xi_2$  are linearly dependent, we begin by noticing that, by definition of  $\varepsilon f_{\#1}$ , we have  $\xi_i \circ \varepsilon f_{\#1} = s\xi_i$ , that is  $\xi_i \in E_{\varepsilon f_{\#1}}^*(s)$ . But, because  $\theta_1$  is an extension of  $\varepsilon f_{\#1}$  to  $S_0(I)$  and  $\theta_1(S_0(I)) \subseteq S_1(I)$ , we have an isomorphism

$$\begin{aligned} \Theta : E_{\varepsilon f_{\#1}}^*(s) &\rightarrow E_{\theta_1}^*(s), \\ \xi &\rightarrow s^{-1}\xi \circ \theta_1. \end{aligned}$$

Since  $\Theta(\xi)|_{S_1(I)} = \xi$ , for all  $\xi \in E_{\varepsilon f_{\#1}}^*(s)$ , we have

$$|\Theta(\xi_i)(x)| = |\Theta(\xi_i)(a) + \Theta(\xi_i)(x - a)| \leq |\Theta(\xi_i)(a)| + b - a$$

for all  $x \in I$ , and from Eq. (15) it follows

$$\Theta(\xi_i) \in B_{\theta_0}(1) \cap E_{\theta_1}^*(s), \quad \text{for } i = 1, 2.$$

Finally, by Eq. (16) and because  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1$ , we may use Theorem 10 to conclude

$$\dim B_{\theta_0}(1) \cap E_{\theta_1}^*(s) = 1,$$

and this shows that  $\Theta(\xi_1)$  and  $\Theta(\xi_2)$  (and consequently  $\xi_1$  and  $\xi_2$ ) are linearly dependent, as desired.  $\square$

## 2.2. The matrix $M(f; y)$

Let  $f \in \mathcal{M}(I)$  and  $y \in I$ . For each  $J = [c, d] \subseteq I$ , let us define the generating function  $\Lambda([c, d]; y; t) \in \mathbb{R}[[t]]$ , that, briefly counts the number of solutions, in  $J$ , of the equations  $f^n(x) = y$ . Define

$$\Lambda([c, d]; y; t) = 1 - \sum_{n \geq 0} \gamma([c, d]; y; n) t^{n+1},$$

with

$$\gamma([c, d]; y; n) = \#\{x \in ]c, d[: f^n(x) = y\} + \frac{1}{2} \#\{x \in \{c, d\}: f^n(x) = y\},$$

and the corresponding radius of convergence

$$\rho(J; y) = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{\gamma(J; y; n)}}.$$

Notice that, since  $\gamma(J; y; n) \leq \ell(f^n)$  for all  $n \geq 0$ , from Eq. (1) we have

$$\rho(J, y) \geq s^{-1}, \quad (19)$$

for all  $J \subseteq I$  and  $y \in I$ . Furthermore, if  $\rho(J; y) < \infty$ , then, for some  $n$ , the equation  $f^n(x) = y$  has at least one solution lying in  $J$ . Thus, as an immediate consequence of the definitions, we obtain the following result, which shows the importance of the numbers  $\rho(J, y)$  for detecting the topological transitivity of a given map  $f \in \mathcal{M}(I)$ .

**Proposition 15.** *Let  $f \in \mathcal{M}(I)$ . If  $\rho(J; y) < \infty$ , for all  $J = [c, d] \subseteq I$ , and all  $y$  lying in some dense subset of  $I$ , then  $f \in \mathcal{T}(I)$ .*

Let  $f \in \mathcal{M}(I)$ ,  $y \in I$  and  $J = [c, d] \subseteq I$ . It was shown in [1] that  $\Lambda([c, d]; y; t)$  can be computed in terms of  $\mathbf{D}(t)$ . Indeed, defining a linear form  $\xi_y : S_1(I) \rightarrow \mathbb{R}$  by

$$\xi_y(w - z) = \gamma([z, w]; y; 0), \quad \text{for all } z, w \in I \text{ with } w > z,$$

from the definition of  $\varepsilon f_{\#1}$ , we have  $\xi_y \circ \varepsilon f_{\#1}(d - c) = \gamma([c, d]; y; 1)$  and  $(\varepsilon f_{\#1})^n = \varepsilon f_{\#1}^n$ , for all  $n \geq 0$ , and thus

$$\xi_y \circ (\varepsilon f_{\#1})^n(d - c) = \xi_y \circ \varepsilon f_{\#1}^n(d - c) = \gamma([c, d]; y; n), \quad \text{for all } n \geq 0. \quad (20)$$

Because  $(\varepsilon f_{\#1}, \varepsilon f_{\#1} + \xi_y \otimes (d - c))$  is, evidently, a pair of linear endomorphisms on  $S_1(I)$  with finite rank, from Eq. (20) and Theorem 9 we see that

$$\Lambda([c, d]; y; t) = D_{(\varepsilon f_{\#1}, \varepsilon f_{\#1} + \xi_y \otimes (d - c))}(t)$$

holds in  $\mathbb{R}[[t]]$ . But on the other hand, by Proposition 8, we also have that

$$\begin{aligned} D_{(\varepsilon f_{\#1}, \varepsilon f_{\#1} + \xi_y \otimes (d - c))}(t) &= D_{(\varepsilon f_{\#1}, \varepsilon f_{\#0})}(t) D_{(\varepsilon f_{\#0}, \varepsilon f_{\#1} + \xi_y \otimes (d - c))}(t) \\ &= D_{(\varepsilon f_{\#0}, \varepsilon f_{\#1})}(t)^{-1} D_{(\varepsilon f_{\#0}, \varepsilon f_{\#1} + \xi_y \otimes (d - c))}(t), \end{aligned}$$

holds in  $\mathbb{R}[[t]]$ . So, if we define

$$\mathbf{D}_{(J; y)}(t) = D_{(\varepsilon f_{\#0}, \varepsilon f_{\#1} + \xi_y \otimes (d - c))}(t) \in \mathbb{R}[[t]],$$

we obtain:

**Proposition 16.** *Let  $f \in \mathcal{M}(I)$ ,  $y \in I$  and  $J = [c, d] \subseteq I$ . Then*

$$\Lambda(J; y; t) = \mathbf{D}_{(J; y)}(t) \mathbf{D}(t)^{-1}$$

*holds in  $\mathbb{R}[[t]]$ .*

Once again we can use Theorem 9 to compute  $\mathbf{D}_{(J; y)}(t)$ . Defining the linear form  $\delta_y : S_0(I) \rightarrow \mathbb{R}$  by

$$\delta_y(x) = \frac{1}{2}(1 + \alpha_y(x)), \quad \text{for all } x \in I,$$

where  $\alpha_y$  is the step function of Eq. (10), we see at once that  $\theta_1 + \delta_y \otimes (d - c)$  is an extension of  $\varepsilon f_{\#1} + \xi_y \otimes (d - c)$  to  $S_0(I)$  verifying  $(\theta_1 + \delta_y \otimes (d - c))(S_0(I)) \subseteq S_1(I)$ . But by Eq. (12)

$$\theta_1 + \delta_y \otimes (d - c) - \theta_0 = \sum_{j=1}^{\ell+1} \omega_j \otimes v_j,$$

with  $v_{\ell+1} = d - c \in S_0(I)$  and  $\omega_{\ell+1} = \delta_y \in S_0(I)^*$ , and from Theorem 9

$$D_{(\varepsilon f_{\#0}, \varepsilon f_{\#1} + \xi_y \otimes (d - c))}(t) = \det(\mathbf{I} - t\mathbf{M}_{([c, d]; y)}(t)) D_{\tilde{\theta}_0}(t)$$

holds in  $\mathbb{R}[[t]]$ , where the matrix  $\mathbf{M}_{([c, d]; y)}(t) = [\mathbf{n}_{i, j}(t)] \in \mathbb{R}[[t]]^{(\ell+1) \times (\ell+1)}$  is defined by:

$$\mathbf{n}_{i, j}(t) = \sum_{n=0}^{\infty} \omega_j(\theta_0^n(v_i)) t^n \in \mathbb{R}[[t]],$$

for all  $1 \leq i, j \leq \ell + 1$ .

Observe that the entries of the matrix  $\mathbf{M}_{([c, d]; y)}(t)$  are formal power series that can be computed in the orbits of points  $f(c_1), \dots, f(c_\ell), c, d$ . Once again, from Eq. (15) and because  $\{\omega_1, \dots, \omega_{\ell+1}\} \subseteq B(2)$ , these formal power series are convergent for all  $|t| < 1$ . So, for any  $|\lambda| > 1$ , we have a matrix  $\mathbf{M}_{([c, d]; y)}(\lambda^{-1}) \in \mathbb{R}^{(\ell+1) \times (\ell+1)}$ , and the proof of the following proposition is similar to the one of Proposition 13.

**Proposition 17.** *Let  $f \in \mathcal{M}(I)$ ,  $J = [c, d] \subseteq I$  and  $y \in I$ . Then the formal power series  $\mathbf{D}_{(J; y)}(t)$  converges for all  $|t| < 1$ . Furthermore, if  $|\lambda| > 1$  then  $\mathbf{D}_{(J; y)}(\lambda^{-1}) = 0$  if and only if  $\lambda$  is an eigenvalue of  $\mathbf{M}_{(J; y)}(\lambda^{-1})$ .*

We have now everything that is necessary to define the matrix  $\mathbf{M}(f; y)$ . Notice that, if  $f \in \mathcal{M}^+(I)$ , we have  $s > 1$ , and therefore one can define the matrix  $\mathbf{M}(f; y) \in \mathbb{R}^{(\ell+1) \times (\ell+1)}$  by setting

$$\mathbf{M}(f; y) = \mathbf{M}_{(I; y)}(s^{-1}),$$

and from Proposition 17

$$s \in \sigma(\mathbf{M}(f; y)) \quad \text{if and only if} \quad \mathbf{D}_{(I; y)}(s^{-1}) = 0. \quad (21)$$

We are now ready to prove Theorem 3. Indeed, from Proposition 15, Theorem 3 follows from the next result:

**Proposition 18.** *Let  $f \in \mathcal{L}(I)$  and  $y \in I$  such that  $s \notin \sigma(\mathbf{M}(f; y))$ . If  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1$ , then  $\rho(J; y) = s^{-1}$  for all  $J = [c, d] \subseteq I$ .*

Proposition 18 is an immediate consequence of the two following lemmas.

**Lemma 19.** *Let  $f \in \mathcal{M}^+(I)$  and  $y \in I$  such that  $s \notin \sigma(\mathbf{M}(f; y))$ . Then  $\rho(I; y) = s^{-1}$ .*

**Proof.** From Propositions 13, 17 and Eqs. (17), (21), the meromorphic function on  $|t| < 1$ , defined by  $\mathbf{D}_{(I;y)}(t)/\mathbf{D}(t)$ , has a pole at  $s^{-1}$ . Therefore, from Proposition 16, we have  $\rho(I; y) \leq s^{-1}$ , and by Eq. (19) it follows  $\rho(I; y) = s^{-1}$ .  $\square$

Remark that if  $f \in \mathcal{M}^+(I)$ , we can have  $\rho(I; y) = s^{-1}$  and  $\rho(J; y) > s^{-1}$  for some interval  $J \subseteq I$ . However, if  $f \in \mathcal{L}(I)$  and  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1$ , then this cannot happen.

**Lemma 20.** Let  $f \in \mathcal{L}(I)$  and  $y \in I = [a, b]$  such that  $\rho(I; y) = s^{-1}$ . If  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1$ , then  $\rho(J; y) = s^{-1}$  for all  $J = [c, d] \subseteq I$ .

**Proof.** Since  $\rho(I; y) = s^{-1}$ , a standard argument (see [7] or [4]) allows us to define a  $s$ -semiconjugacy of  $f$  by setting

$$h(x) = a + \lim_{t \rightarrow s^{-1}} \frac{(b-a)\Lambda([a, x]; y; t)}{\Lambda(I; y; t)},$$

for all  $x \in I$ . Suppose that  $\rho(J; y) > s^{-1}$ , for some  $J = [c, d] \subseteq I$ . In this case the same arguments also show that  $h$  is constant on  $J$ , and therefore  $h$  is not an homeomorphism. But this is a contradiction because from Theorem 14 there exists one and only one  $s$ -semiconjugacy of  $f$  which must be an homeomorphism because  $f \in \mathcal{L}(I)$ .  $\square$

In order to improve the computational aspects of Theorem 3, let us notice that the entry  $(\ell+1, \ell+1)$  of the matrix  $\mathbf{M}(f; y)$  is completely irrelevant to have  $s \notin \sigma(\mathbf{M}(f; y))$ . Let  $f \in \mathcal{M}^+(I)$ , with  $I = [a, b]$ . In what follows, and in order to simplify the notations, we shall denote the entries of  $\mathbf{M}(f)$  by  $\mathbf{m}_{i,j}$ , that is

$$\mathbf{m}_{i,j} = \sum_{n \geq 0} \omega_j(\theta_0^n(f_{\#0}(c_i)))s^{-n}, \quad (22)$$

for all  $1 \leq i, j \leq \ell$ . Since the entries of the last column of  $\mathbf{M}(f; y)$  depend on  $y$  will be denoted by  $\mathbf{n}_i(y)$ . The entries of the last row of  $\mathbf{M}(f; y)$  (except the last one) will be denoted by  $\mathbf{n}_{\ell+1,j}$ . So, with this notations, we can write:

$$\mathbf{M}(f; y) = \left[ \begin{array}{c|c} \mathbf{M}(f) & \begin{matrix} \mathbf{n}_1(y) \\ \vdots \\ \mathbf{n}_\ell(y) \end{matrix} \\ \hline \mathbf{n}_{\ell+1,1} & \dots & \mathbf{n}_{\ell+1,\ell} & \mathbf{n}_{\ell+1}(y) \end{array} \right], \quad (23)$$

with

$$\mathbf{n}_{\ell+1,i} = \sum_{n=0}^{\infty} \omega_i(\theta_0^n(b-a))s^{-n} \quad \text{and} \quad \mathbf{n}_i(y) = \sum_{n=0}^{\infty} \delta_y(\theta_0^n(f_{\#0}(c_i)))s^{-n}, \quad (24)$$

for all  $1 \leq i \leq \ell$ , and

$$\mathbf{n}_{\ell+1}(y) = \sum_{n=0}^{\infty} \delta_y(\theta_0^n(b-a))s^{-n}.$$

Notice that, if we define the map  $\mathbf{t}: I \rightarrow \mathbb{R}$  (not depending on  $\mathbf{n}_{\ell+1}(y)$ ) by

$$\mathbf{t}(y) = \sum_{i=1}^{\ell} (-1)^{i+\ell+1} \det(\mathbf{N}_i) \mathbf{n}_i(y), \quad (25)$$

for all  $y \in I$ , where  $\mathbf{N}_i \in \mathbb{R}^{\ell \times \ell}$ , is the matrix obtained from  $\mathbf{M}(f) - s\mathbf{I}$ , replacing its  $i$ -row by  $[\mathbf{n}_{\ell+1,1} \ \dots \ \mathbf{n}_{\ell+1,\ell}]$ , then by Eq. (23) and using Laplace rule

$$\det(\mathbf{M}(f; y) - s\mathbf{I}) = \mathbf{n}_{\ell+1}(y) \det(\mathbf{M}(f) - s\mathbf{I}) + \mathbf{t}(y),$$

and from Eq. (18)

$$\det(\mathbf{M}(f; y) - s\mathbf{I}) = \mathbf{t}(y),$$

which shows that  $s \notin \sigma(\mathbf{M}(f; y))$  if and only if  $\mathbf{t}(y) \neq 0$ . So as an immediate consequence of Theorem 3, we obtain the following:

**Corollary 21.** *Let  $f \in \mathcal{L}(I)$ . If  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1$  and  $\mathbf{t}(y) \neq 0$ , for all  $y$  lying in some dense subset of  $I$ , then  $f \in \mathcal{T}(I)$ .*

**Example 22.** For each  $q \in ]0, 1/2[$  and  $s \in ](1-q)^{-1}, q^{-1}[$  let  $f_{q,s}: [0, 1] \rightarrow [0, 1]$  be the  $s$ -piecewise linear map, defined by

$$f_{q,s}(x) = \begin{cases} s(q-x) & 0 \leq x \leq q, \\ s(x-q) & q \leq x \leq q+s^{-1}, \\ 2+s(q-x) & q+s^{-1} \leq x \leq 1. \end{cases}$$

The map  $f_{q,s}$  has exactly two turning points which are  $c_1 = q$  and  $c_2 = q + s^{-1}$ . We have  $f_{q,s}(c_1) = 0$ ,  $f_{q,s}(c_2) = 1$ , and by Eq. (11), the linear forms  $\omega_1$  and  $\omega_2$  are defined by:

$$\omega_1(x) = \begin{cases} -1 & \text{if } x > c_1, \\ 0 & \text{if } x = c_1, \\ 1 & \text{if } x < c_1, \end{cases} \quad \text{and} \quad \omega_2(x) = \begin{cases} 2 & \text{if } x > c_2, \\ 1 & \text{if } x = c_2, \\ 0 & \text{if } x < c_2, \end{cases}$$

for all  $x \in I$ . For  $s = 2$ , and  $q \in ]0, 1/2[$  the map  $f_{q,2}$  is not topologically transitive because  $f_{q,2}([0, 2q]) = [0, 2q]$ . But this does not contradict Theorem 3. In fact since  $f_{q,2}(0) = f_{q,2}(1) = f_{q,2}^n(c_1) = f_{q,2}^n(c_2) = 2q$ , for all  $n \geq 1$ , a simple calculation shows that

$$M(f_{q,2}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and thus  $\dim \ker(\mathbf{M}(f_{q,2}) - 2\mathbf{I}) = 2$ . We can use the same argument to prove that  $f_{q,s}$  is not topologically transitive if  $1 < s \leq 2$ . In fact, for a such map we have  $f_{q,s}([0, 2q]) = [0, s q] \subseteq [0, 2q]$ .

Now we assume that  $s > 2$ . In this case we have a completely different situation. Let us begin by noticing that in this case we have

$$\dim \ker(\mathbf{M}(f_{q,s}) - s\mathbf{I}) = 1.$$

Actually, if  $\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 2$ , then (since  $M(f_{q,s})$  is a  $2 \times 2$  matrix) we would have

$$M(f_{q,s}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix},$$

but this is a contradiction because  $\mathbf{m}_{2,1}$  is different from zero. This is an immediate consequence of the definitions: from Eq. (22), we have

$$\mathbf{m}_{2,1} = \sum_{n=0}^{\infty} \omega_1(\theta_0^n(1))s^{-n} = -1 + \sum_{n=1}^{\infty} \omega_1(\theta_0^n(1))s^{-n},$$

since  $\omega_1(\theta_0^n(1)) \in \{-1, 0, 1\}$ , for all  $n$ , and because  $s > 2$ , it follows

$$\left| \sum_{n=1}^{\infty} \omega_1(\theta_0^n(1))s^{-n} \right| \leq \sum_{n=1}^{\infty} s^{-n} < 1,$$

and therefore  $\mathbf{m}_{2,1} < 0$ , as desired. Next we study the map  $\mathbf{t}: I \rightarrow \mathbb{R}$ . Notice that, since  $f_{q,s}(c_1) = 0$  and  $f_{q,s}(c_2) = 1$ , simple computations show that:

$$\det(\mathbf{N}_1) = s(\mathbf{m}_{2,1} + \mathbf{m}_{2,2} - s) \quad \text{and} \quad \det(\mathbf{N}_2) = s(\mathbf{m}_{1,1} + \mathbf{m}_{1,2} - s),$$

where  $\mathbf{N}_1$  and  $\mathbf{N}_2$  are the matrices defined in Eq. (25). We have then:

$$\mathbf{t}(y) = s(\mathbf{m}_{2,1} + \mathbf{m}_{2,2} - s)\mathbf{n}_1(y) - s(\mathbf{m}_{1,1} + \mathbf{m}_{1,2} - s)\mathbf{n}_2(y),$$

for all  $y \in I$ . Remark that, defining

$$\mathbf{m}_{i,j}^{(k)} = \sum_{n=0}^k \omega_j(\theta_0^n(f_{\#0}(c_i)))s^{-n} \quad \text{and} \quad \mathbf{n}_i^{(k)}(y) = \sum_{n=0}^k \delta_y(\theta_0^n(f_{\#0}(c_i)))s^{-n},$$

simple computations show that

$$|\mathbf{t}(y) - \mathbf{t}_k(y)| \leq \frac{18}{s^{k-2}}, \quad (26)$$

where  $\mathbf{t}_k: I \rightarrow \mathbb{R}$  is the step function defined by

$$\mathbf{t}_k(y) = s(\mathbf{m}_{2,1}^{(k)} + \mathbf{m}_{2,2}^{(k)} - s)\mathbf{n}_1^{(k)}(y) - s(\mathbf{m}_{1,1}^{(k)} + \mathbf{m}_{1,2}^{(k)} - s)\mathbf{n}_2^{(k)}(y).$$

By definition, this step function just depends on the finite set

$$O_k = \{f_{q,s}^n(c_i): i = 1, 2 \text{ and } 1 \leq n \leq k+1\}.$$

Since the set of all discontinuities of  $\mathbf{t}_k$  is contained in  $O_k$ , from Eq. (26), we obtain the following: if

$$\min\{|\mathbf{t}_k(y^-)|: y \in O_k\} > \frac{18}{s^{k-2}},$$

then  $\mathbf{t}(y) \neq 0$ , for all  $y \in I \setminus O_k$ , and from Corollary 21  $f_{q,s} \in \mathcal{T}(I)$ . As an example consider the map  $f_{q,s}$  with  $q = 0.3$  and  $s = e$ . Notice that, since  $e$  is transcendent, at least one of the orbits of its turning points is infinite, but this does not disable us of proving that  $f_{q,s} \in \mathcal{T}(I)$ . Actually it is enough to know  $O_4$  to conclude

$$\min\{|\mathbf{t}_k(y^-)|: y \in O_4\} = 5.8 \dots > \frac{18}{e^2},$$

and consequently  $f_{q,s} \in \mathcal{T}(I)$ .

### 2.3. Proof of Theorem 4

Let  $c_1 < \dots < c_\ell$  be the turning points of a Markov map  $f$ . By definition, the set

$$P = \{f^n(c_i) : n \geq 0 \text{ and } 1 \leq i \leq \ell\}$$

is finite, and consequently the vector spaces  $S_0(P)$  and  $S_1(P)$  are finite-dimensional. By definition of  $\varepsilon f_{\#1}$ ,  $\theta_0$  and  $\theta_1$  we have

$$\varepsilon f_{\#1}(S_1(P)) \subseteq S_1(P) \quad \text{and} \quad \theta_i(S_0(P)) \subseteq S_0(P), \quad \text{for } i = 0, 1,$$

and so, we may consider the linear endomorphisms

$$\varepsilon f_{\#1} : S_1(P) \rightarrow S_1(P) \quad \text{and} \quad \theta_i : S_0(P) \rightarrow S_0(P), \quad \text{for } i = 0, 1,$$

and a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_1(P) & \xrightarrow{\subseteq} & S_0(P) & \xrightarrow{\text{pr}} & S_0(P)/S_1(P) \longrightarrow 0 \\ & & \downarrow \varepsilon f_{\#1} & & \downarrow \theta_1 & & \downarrow 0 \\ 0 & \longrightarrow & S_1(P) & \xrightarrow{\subseteq} & S_0(P) & \xrightarrow{\text{pr}} & S_0(P)/S_1(P) \longrightarrow 0 \end{array} \quad (27)$$

By Eq. (12) we also have

$$\theta_1 - \theta_0 = \sum_{j=1}^{\ell} \omega_j \otimes f_{\#0}(c_j),$$

where  $\omega_j \in S_0(P)^*$  denotes the restriction to  $S_0(P)$  of the linear forms defined in Eq. (11). Notice that, if we denote the points of  $P$  by  $p_0 < \dots < p_k$ , then the set

$$\{p_1 - p_0, p_2 - p_1, \dots, p_k - p_{k-1}\}$$

is a basis of  $S_1(P)$ , and the matricial representation of  $\varepsilon f_{\#1}$  with respect to this basis coincides with the transition matrix of  $f$ ,  $\mathbf{A} = [a_{i,j}]_1^k$ , defined by:

$$a_{i,j} = \begin{cases} 1 & \text{if } [p_{i-1}, p_i] \subseteq f([p_{j-1}, p_j]), \\ 0 & \text{otherwise.} \end{cases}$$

This enables us to prove the following:

**Lemma 23.** *Let  $f \in \mathcal{T}(I)$  be a Markov map. Then  $s^{-1}$  is a simple zero of  $\mathbf{D}(t)$ .*

**Proof.** Since  $f \in \mathcal{T}(I)$ , the matrix  $\mathbf{A}$  is irreducible and from Perron–Frobenius Theorem, we know that  $s^{-1}$  is a simple zero of  $\det(\mathbf{I} - t\mathbf{A})$ . On the other hand, from Eq. (4)

$$D_{\varepsilon f_{\#1}}(t) = \det(\mathbf{I} - t\mathbf{A})$$

holds in  $\mathbb{R}[[t]]$ , and from the commutative diagram of Eq. (27)

$$D_{\varepsilon f_{\#1}}(t) = D_{\theta_1}(t) = D_{\theta_1}(t) D_{\theta_0}(t)^{-1} D_{\theta_0}(t).$$

Thus, from Eq. (6) and Theorem 9

$$\det(\mathbf{I} - t\mathbf{A}) = \det(\mathbf{I} - t\mathbf{M}(t)) D_{\theta_0}(t),$$



and by Eq. (13) it follows that

$$\det(\mathbf{I} - t\mathbf{A})D_{\tilde{\theta}_0}(t)D_{\theta_0}(t)^{-1} = \mathbf{D}(t)$$

holds in  $\mathbb{R}[[t]]$ . Finally, because  $D_{\tilde{\theta}_0}(s^{-1}) \neq 0$  (see the proof of Proposition 13) and  $D_{\theta_0}(t)$  is a polynomial, it follows that  $s^{-1}$  is a simple zero of  $\mathbf{D}(t)$ .  $\square$

**Lemma 24.** *Let  $f \in \mathcal{T}(I)$  be a Markov map. Then  $\rho(I, y) = s^{-1}$ , for all  $y \in I$ .*

**Proof.** Let  $\mathbf{A}$  be the transition matrix of  $f$ . Notice that, since

$$\lim_n \sqrt[n]{\text{tr}(\mathbf{A}^n)} = s,$$

if we denote the entry  $(i, j)$  of  $\mathbf{A}^n$  by  $a_{i,j}^{(n)}$ , we have  $\text{tr}(\mathbf{A}^n) = a_{1,1}^{(n)} + \dots + a_{k,k}^{(n)}$ , and therefore

$$\lim_n \sqrt[n]{a_{i,i}^{(n)}} = s,$$

for some  $i = 1, \dots, k$ . We this we conclude that  $\rho(I, y) = s^{-1}$ , for all  $y \in I_i$ . But, by the definition of  $\rho(I, y)$ , if  $\rho(I, y) = s^{-1}$ , for all  $y \in I_i$ , then  $\rho(I, y) = s^{-1}$ , for all  $y \in f^n(I_i)$  and all  $n \geq 1$ , and because  $f \in \mathcal{T}(I)$ , it follows  $\rho(I, y) = s^{-1}$ , for all  $y \in I$ , as desired.  $\square$

We are now ready to prove the following:

**Proposition 25.** *Let  $f \in \mathcal{T}(I)$  be a Markov map. Then  $s \notin \sigma(\mathbf{M}(f; y))$ , for all  $y \in I$ .*

**Proof.** Let  $y \in I$ . From Lemma 24 and Proposition 16, the meromorphic function  $\mathbf{D}_{(I;y)}(t)/\mathbf{D}(t)$  has a pole lying in  $|t| = s^{-1}$ , and consequently

$$\sup\{|\Lambda(I; y; t)| : |t| < s^{-1}\} = \infty.$$

But on the other hand, since  $\gamma(I; y; n) \geq 0$ , for all  $n \geq 0$ , we also have

$$\begin{aligned} |\Lambda(I; y; t)| &= \left| 1 - \sum_{n \geq 0} \gamma(I; y; n) t^{n+1} \right| \leq 1 + \sum_{n \geq 0} \gamma(I; y; n) |t|^{n+1} \\ &= 2 - \Lambda(I; y; |t|), \end{aligned}$$

for all  $|t| < s^{-1}$ , and thus

$$\sup\{|\Lambda(I; y; |t|)| : |t| \in [0, s^{-1}[ ] = \infty.$$

This shows that  $\mathbf{D}_{(I;y)}(t)/\mathbf{D}(t)$  has a pole at  $s^{-1}$ , but from Lemma 23, it follows  $\mathbf{D}_{(I;y)}(s^{-1}) \neq 0$ , and by Eq. (21)  $s \notin \sigma(\mathbf{M}(f; y))$  as desired.  $\square$

To prove Theorem 4, it remains to prove the following:

**Proposition 26.** *Let  $f \in \mathcal{T}(I)$  be a Markov map. Then*

$$\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = 1.$$

**Proof.** Since the spectral radius of  $\mathbf{A}^T$  is precisely  $s$ , and  $\mathbf{A}^T$  is irreducible, from Perron–Frobenius Theorem we have  $\dim \ker(\mathbf{A}^T - s\mathbf{I}) = 1$ , and thus

$$\dim \ker(\varepsilon f_{\#1}^* - s\mathbf{I}) = \dim \ker(\mathbf{A}^T - s\mathbf{I}) = 1.$$

On the other hand, because  $\theta_1$  is an extension of  $\varepsilon f_{\#1}$  to  $S_0(P)$ , verifying  $\theta_1(S_0(P)) \subseteq S_1(P)$ , the argument used in the proof of Theorem 14 shows that

$$\dim \ker(\theta_1^* - s\mathbf{I}) = \dim \ker(\varepsilon f_{\#1}^* - s\mathbf{I}) = 1.$$

Finally, since  $s > r(\theta_0) = 1$ , from Corollary 11, we have

$$\dim \ker(\mathbf{M}(f) - s\mathbf{I}) = \dim \ker(\theta_1^* - s\mathbf{I}) = 1,$$

as desired.  $\square$

## References

- [1] J.F. Alves, J. Sousa Ramos, Kneading theory: A functorial approach, *Comm. Math. Phys.* 204 (1) (1999) 89–114.
- [2] J.F. Alves, J. Sousa Ramos, One-dimensional semiconjugacy revisited, *Internat. J. Bifur. Chaos* 13 (7) (2003) 1657–1663.
- [3] A. Blokh, On sensitive mappings of the interval, *Russian Math. Surveys* 37 (2) (1982) 203–204.
- [4] J. Milnor, W. Thurston, On iterated maps of the interval, in: *Dynamical Systems: Proceedings 1986–1987*, in: *Lecture Notes in Math.*, vol. 1342, Springer, Berlin, 1988, pp. 465–563.
- [5] M. Misiurewicz, W. Szlenk, Entropy of piecewise monotone mappings, *Astérisque* 50 (1977) 299–310 (short version); *Studia Math.* 67 (1980) 45–63.
- [6] W. Parry, Symbolic dynamics and transformations of unit interval, *Trans. Amer. Math. Soc.* 122 (1964) 368–378.
- [7] C. Preston, What you need to knead, *Adv. in Math.* 78 (1989) 192–252.